

# PÓLYA'S RANDOM WALK THEOREM

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This note is about a remarkable law of nature discovered by George Pólya [6]. Consider a particle situated at a given point of the integer lattice  $\mathbb{Z}^d$ . Suppose that, at each tick of the clock, the particle jumps to a randomly selected neighbouring lattice point, with equal probability of jumping in any direction. In other words, this particle is executing the *simple random walk* on  $\mathbb{Z}^d$ .

A random walk is said to be *recurrent* if it returns to its initial position with probability one. A random walk which is not recurrent is called *transient*. Pólya's classic result is the following.

**Theorem 1.** *The simple random walk on  $\mathbb{Z}^d$  is recurrent in dimensions  $d = 1, 2$  and transient in dimension  $d \geq 3$ .*

This note presents a fairly direct proof of Pólya's Theorem which differs from both the standard probabilistic proof (see e.g. [4, Chapter 4]) and the approach via electrical networks [3]. The argument presented here combines basic methods from combinatorics (decompositions and generating functions), special functions (Bessel function identities), quantum field theory (Borel transform), and asymptotic analysis (Laplace's method). Since it illustrates these broadly applicable techniques working in tandem to establish a foundational result of probability theory, it is of pedagogical value.

## 1. LOOP DECOMPOSITION

Let  $E$  denote the event that the simple random walk on  $\mathbb{Z}^d$  returns to its initial position, and put  $p = \text{Prob}(E)$ . For  $n \geq 1$ , let  $E_n$  be the event that the random walk returns to its initial position for the first time after  $n$  steps. It is convenient to set  $E_0 = \emptyset$ , corresponding to the fact that the initial position of the random walk does not count as a return (if it did, the return probability of any random walk would be one). The events  $E_n$  are mutually exclusive for different values of  $n$ , and

$$E = \bigsqcup_{n \geq 0} E_n.$$

Hence

$$p = \sum_{n \geq 0} p_n,$$

where  $p_n = \text{Prob}(E_n)$ .

A *loop* on  $\mathbb{Z}^d$  is a walk which begins and ends at a given point. It is convenient to consider walks of length zero as loops; such loops are called *trivial*. A non-trivial loop is *indecomposable* if it is not the concatenation of two non-trivial loops. Choose a particular point of  $\mathbb{Z}^d$ , and let  $\ell_n$  denote the number of loops of length  $n$  based at this point. Let  $r_n$  denote the number of these which are indecomposable. Note

that  $\ell_0 = 1$  while  $r_0 = 0$ . Since any non-trivial loop is the concatenation of an indecomposable loop followed by a (possibly trivial) loop, the counts  $\ell_n$  and  $r_n$  are related by

$$\ell_n = \sum_{k=0}^n r_k \ell_{n-k}$$

for all  $n \geq 1$ . Dividing both sides of this equation by  $(2d)^n$ , we obtain the relation

$$q_n = \sum_{k=0}^n q_k p_{n-k}$$

for all  $n \geq 1$ , where as above  $p_n$  is the probability that the random walk returns to its initial position for the first time after  $n$  steps, while  $q_n$  is the probability that the random walk is located at its original position after  $n$  steps.

Introduce the generating functions

$$P(z) = \sum_{n=0}^{\infty} p_n z^n \quad \text{and} \quad Q(z) = \sum_{n=0}^{\infty} q_n z^n.$$

The relation between  $p_n$  and  $q_n$  is then equivalent to

$$P(z)Q(z) = Q(z) - 1.$$

Since  $p_n \leq q_n \leq 1$ , each of these series has radius of convergence at least one. Thus we may consider  $P(z), Q(z)$  as analytic functions defined on the open unit disc in the complex plane. The function  $Q(z)$  is non-vanishing for  $z$  in the interval  $[0, 1)$ , and hence we have

$$P(z) = 1 - \frac{1}{Q(z)}, \quad z \in [0, 1).$$

Since

$$P(1) = \sum_{n=0}^{\infty} p_n = p,$$

we have

$$p = 1 - \frac{1}{\sum_{n=0}^{\infty} q_n}.$$

Thus Pólya's Theorem is equivalent to the statement that the sum  $Q(1)$  diverges for  $d = 1, 2$  and converges for  $d \geq 3$ .

## 2. EXPONENTIAL LOOP GENERATING FUNCTION

In order to analyze the sum  $Q(1)$ , we will obtain a tractable representation of the function  $Q(z)$ . This amounts to finding an expression for the loop generating function

$$L(z) = \sum_{n=0}^{\infty} \ell_n z^n.$$

Indeed,  $Q(z) = L(\frac{z}{2d})$ .

While the ordinary generating function  $L(z)$  is difficult to analyze directly, the exponential loop generating function

$$E(z) = \sum_{n=0}^{\infty} \ell_n \frac{z^n}{n!}$$

is quite accessible. This is because any loop on  $\mathbb{Z}^d$  is a shuffle of loops on  $\mathbb{Z}^1$ , and products of exponential generating functions correspond to shuffles. This is a basic property of exponential generating functions which we will review in the specific case at hand. For a general treatment, the reader is referred to [7, Chapter 5].

In this paragraph it is important to make the dependence on  $d$  explicit, so we write  $\ell_n^{(d)}$  for the number of length  $n$  loops on  $\mathbb{Z}^d$  and  $E_d(z)$  for the exponential generating function of this sequence. Let us consider the case  $d = 2$ . A loop on  $\mathbb{Z}^2$  is a closed walk which takes unit steps in two directions, horizontal and vertical. A length  $n$  loop on  $\mathbb{Z}^2$  is made up of some number  $k$  of horizontal steps together with  $n - k$  vertical steps. The  $k$  horizontal steps constitute a length  $k$  loop on  $\mathbb{Z}$ , and the  $n - k$  vertical steps constitute a length  $n - k$  loop on  $\mathbb{Z}$ . Thus the number of length  $n$  loops on  $\mathbb{Z}^2$  which take  $k$  horizontal and  $n - k$  vertical steps is

$$\binom{n}{k} \ell_k^{(1)} \ell_{n-k}^{(1)},$$

since specifying the times at which the  $k$  horizontal steps occur uniquely determines the times at which the  $n - k$  vertical steps occur. The total number of length  $n$  loops on  $\mathbb{Z}^2$  is therefore

$$\ell_n^{(2)} = \sum_{k=0}^n \binom{n}{k} \ell_k^{(1)} \ell_{n-k}^{(1)}.$$

This is equivalent to the generating function identity

$$E_2(z) = E_1(z)^2.$$

The same reasoning applies for any  $d$ , and in general we have

$$E_d(z) = E_1(z)^d.$$

### 3. BESSEL FUNCTIONS

Counting loops in one dimension is easy:

$$\ell_n^{(1)} = \begin{cases} \binom{2k}{k}, & \text{if } n = 2k \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}.$$

Indeed, any loop on  $\mathbb{Z}$  consists of  $k$  positive steps and  $k$  negative steps for some  $k \geq 0$ , and the times at which the positive steps occur determine the times at which the negative steps occur. Thus

$$E_1(z) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{z^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!k!}.$$

Now a minor miracle occurs: the exponential generating function for lattice walks in one dimension is a *modified Bessel function of the first kind*.

The modified Bessel function of the first kind, usually denoted  $I_\alpha(z)$ , is one of two linearly independent solutions to the second order differential equation

$$\left(z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} - (z^2 + \alpha^2)\right) F(z) = 0, \quad \alpha \in \mathbb{C}.$$

This differential equation is known as the *modified Bessel equation*; it appears in a multitude of physical problems, and was exhaustively studied by nineteenth century mathematicians. An excellent reference on this subject is [1, Chapter 4]. It is known that the modified Bessel function admits both a series representation,

$$I_\alpha(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+\alpha}}{k! \Gamma(k + \alpha + 1)},$$

and an integral representation,

$$I_\alpha(z) = \frac{\left(\frac{z}{2}\right)^\alpha}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_0^\pi e^{(\cos \theta)z} (\sin \theta)^{2\alpha} d\theta.$$

From the series representation, we see that  $E_1(z) = I_0(2z)$ , and hence

$$E(z) = I_0(2z)^d.$$

#### 4. BOREL TRANSFORM

We now have a representation of the exponential generating function  $E(z)$  of loops on  $\mathbb{Z}^d$  in terms of a standard mathematical object, the modified Bessel function  $I_0(z)$ . What we need, however, is a representation of the ordinary loop generating function  $L(z)$ .

The integral transform

$$(\mathcal{B}f)(z) = \int_0^\infty f(tz) e^{-t} dt,$$

which looks like the Laplace transform of  $f$  but with the  $z$ -parameter in the wrong place, converts exponential generating functions into ordinary generating functions. To see why, write out the Maclaurin series of  $f(tz)$ , interchange integration and summation to obtain

$$(\mathcal{B}f)(z) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{z^n}{n!} \int_0^\infty t^n e^{-t} dt,$$

and use the fact that

$$\int_0^\infty t^n e^{-t} dt = n!.$$

This trick is standard issue in quantum field theory, where it is constantly used in connection with Borel summation of divergent series, see [5, §2.3].

Applying the Borel transform to our problem, we obtain

$$L(z) = \mathcal{B}E(z) = \mathcal{B}I_0(2z)^d = \int_0^\infty I_0(2tz)^d e^{-t} dt.$$

Thus we have arrived at the integral representation

$$Q(z) = L\left(\frac{z}{2d}\right) = \int_0^\infty I_0\left(\frac{tz}{d}\right)^d e^{-t} dt,$$

from which we obtain

$$Q(1) = \int_0^\infty I_0\left(\frac{t}{d}\right)^d e^{-t} dt.$$

It now remains to prove that this integral diverges for  $d = 1, 2$  and converges for  $d \geq 3$ .

## 5. THE LAPLACE PRINCIPLE

The divergence or convergence of  $Q(1)$  depends only on the behaviour of the tail integral

$$\int_N^\infty I_0\left(\frac{t}{d}\right)^d e^{-t} dt, \quad N \gg 0.$$

This tail behaviour is in turn determined by the behaviour of the integrand as  $t \rightarrow \infty$ .

In order to estimate the integrand, we invoke the integral representation of the modified Bessel function stated above. This yields

$$I_0\left(\frac{t}{d}\right) = \frac{1}{\pi} \int_0^\pi e^{tf(\theta)} d\theta,$$

where  $f(\theta) = \frac{1}{d} \cos \theta$ , and we can estimate this integral as  $t \rightarrow \infty$  using a basic technique of asymptotic analysis known as *Laplace's method*.

The function  $f(\theta)$  is strictly maximized over the interval  $[0, \pi]$  at the left endpoint  $\theta = 0$ . Thus the integrand  $e^{tf(\theta)}$  is exponentially larger at  $\theta = 0$  than at any other point of this interval. As  $t \rightarrow \infty$  this effect becomes increasingly exaggerated, so much so that the integral “localizes” at  $\theta = a$  in the  $t \rightarrow \infty$  limit. To quantify this, note that  $f'(0) = 0$ ,  $f''(0) < 0$ , and consider the quadratic Taylor approximation of  $f(\theta)$ :

$$f(\theta) \approx f(0) - |f''(0)| \frac{\theta^2}{2}.$$

Replacing  $f(\theta)$  with its quadratic approximation, we obtain the integral approximation

$$\int_0^\pi e^{tf(\theta)} d\theta \approx e^{tf(0)} \int_0^\pi e^{-t|f''(0)| \frac{\theta^2}{2}} d\theta.$$

Extending the integral on the right over the positive reals results in a half a Gaussian integral, which can be computed exactly:

$$\int_0^{+\infty} e^{-t|f''(0)| \frac{\theta^2}{2}} d\theta = \sqrt{\frac{\pi}{2t|f''(0)|}}.$$

Thus we expect that

$$\int_0^\pi e^{tf(\theta)} d\theta \approx e^{tf(0)} \sqrt{\frac{\pi}{2t|f''(0)|}}$$

is an approximation of our integral whose accuracy increases as  $t \rightarrow \infty$ . Laplace's principle [2, §5.2] ensures that this is indeed the case:

$$\int_0^\pi e^{tf(\theta)} d\theta \sim e^{tf(0)} \sqrt{\frac{\pi}{2t|f''(0)|}}, \quad t \rightarrow \infty,$$

where the notation  $F(t) \sim G(t)$ ,  $t \rightarrow \infty$  means that  $\lim_{t \rightarrow \infty} \frac{F(t)}{G(t)} = 1$ .

Putting everything together, we have the asymptotic formula

$$I_0 \left( \frac{t}{d} \right)^d e^{-t} \sim \text{constant} \cdot t^{-\frac{d}{2}}, \quad t \rightarrow \infty.$$

Thus we conclude that the recurrence or transience of the simple random walk on  $\mathbb{Z}^d$  is equivalent to the divergence or convergence of the integral

$$\int_N^\infty t^{-d/2} dt, \quad N \gg 0.$$

Since this integral diverges for  $d = 1, 2$  and converges for  $d \geq 3$ , Pólya's Theorem is proved.

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